

On statistical indistinguishability of complete and incomplete discrete time market models

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Abstract

We investigate the possibility of statistical evaluation of the market completeness for discrete time stock market models. It is known that the market completeness is not a robust property: small random deviations of the coefficients convert a complete market model into a incomplete one. The paper shows that market incompleteness is also non-robust. We show that, for any incomplete market from a wide class of discrete time models, there exists a complete market model with arbitrarily close stock prices. This means that incomplete markets are indistinguishable from the complete markets in the terms of the market statistics.

Key words: price statistics, market completeness, market incompleteness, forecasting

JEL classification: C18, C52, C53, G13

1 Introduction

The paper studies discrete time stock market models and their completeness or incompleteness. For the so-called complete market, any claim can be replicated. The classical discrete time Cox-Ross-Rubinstein model of a single-stock financial market is complete; this is a binomial model. For incomplete market models, the option replication is not always possible. Unfortunately, the market completeness is not a robust property: small random deviations can ruin the completeness and convert a complete model into a incomplete one.

In the present paper, we show that the market incompleteness is also non-robust. It appears that, for any incomplete market model from a wide class of models, there exists a complete market model with an arbitrarily close stock prices, in a setting where the admissible portfolio strategies can use historical observations collected before the launching time of the replicating strategy

(Theorem 2.1). In other word, the incomplete markets are indistinguishable from the complete markets in the terms of the market statistics (Corollary 2.1). Arbitrarily small rounding errors and time discretization errors may lead to different market models with respect to the completeness and incompleteness. This contradicts to a common perception that the incompleteness can be spotted from the statistics.

Theorem 2.1 establishes some limits for analysis of market structures based solely on econometrics and provides one more illustration of importance of the agents' beliefs in interpretations of econometrical data, in the framework of the concept from [18]-[19]. Another curious consequence is that the option prices are not robust with respect to small deviations of the past stock prices, since pricing formulas for complete and incomplete models are different (in fact, prices are not uniquely defined for the incomplete market).

Some non-robustness of certain market properties (more precisely, arbitrage opportunities) was considered in [15]. We study a different market property: the incompleteness caused by non-hedgeable randomness of parameters. The arbitrage possibility or completeness are some extreme and rare features. The arbitrage possibility is usually caused by abnormally vanishing volatility or fast growing appreciation rate; the completeness is caused by the predictability and the absence of the noise for the volatility. On the other hand, the incompleteness is rather a typical feature. It is easier to believe that a noise contamination of a model removes some rare property. Hence the result of the present paper is more counterintuitive.

Related results were obtained in [12, 13] and presented by the author on The Quantitative Methods in Finance conference in Sydney in 2013. In [12], diffusion continuous time models were considered; in [13], discrete time high frequency binomial models and their were considered. The result of the present paper was obtained by a different approach.

2 The result

2.1 The market model

Assume that we are given a probability space with a complete σ -algebra of events \mathcal{F} and a probability measure \mathbf{P} . Let \mathbb{Z} be the set of all integers, and let $\mathbb{Z}^- = \{0, -1, -2, -3, \dots\}$.

Consider discrete time model of a securities market consisting of a risky stock with the price $S(t) > 0$ and risk free bond or bank account with the price $B(t)$, for integers t . The process $B(t)$ is assumed to be non-random and such that $B(t) > 0$ a.s. For simplicity, we assume that $B(t+1)/B(t) = \rho$ for some $\rho \geq 1$. Let $\tilde{S}(t) = B(t)^{-1}S(t)$ be the discounted price process. In this

setting, the process $B(t)$ is assumed to be non-random or risk-free and is used as a numéraire.

Let $\{\mathcal{F}_t\}$ be the filtration generated by the flow of observable data, i.e., by the process $S(t)$.

Let $\xi(t) = (\tilde{S}(t)/\tilde{S}(t-1) - 1)$. Clearly, $\tilde{S}(t) = \tilde{S}(t-1)(1 + \xi(t))$.

We assume that $\xi(t) \in (-1, 1)$. It can be noted that the presence of the upper boundary for $\xi(t)$ is actually restrictive since it excludes some important models; however, our proof for the results given below depends on this assumptions.

We assume that there exists a probability measure \mathbf{P}_* being equivalent to \mathbf{P} such that the process $\tilde{S}(t)$ is a martingale with respect to $\{\mathcal{F}_t\}$. Let \mathbf{E}_* be the corresponding expectation.

Let $s, \theta \in \mathbb{Z}$ be given, $s < \theta$. Let $X(t)$ be the wealth at time t and such that

$$X(t) = \beta(t)B(t) + \gamma(t)S(t), \quad t = s, s+1, \dots, \theta, \quad (1)$$

where $\beta(t)$ is the quantity of the bond portfolio and where $\gamma(t)$ is the vector describing the quantities of the shares of the stock portfolio. The pair $(\beta(t), \gamma(t))$ describes the state of the bond-stocks securities portfolio at time t . We call the sequences of these pairs portfolio strategies.

Some constraints will be imposed on current operations in the market.

A portfolio strategy $\{(\beta(t), \gamma(t))\}_{t=s}^\theta$ is said to be admissible and self-financing if the following conditions are satisfied.

- (i) There exists a \mathbf{P} -equivalent martingale measure \mathbf{P}_* such that $\mathbf{E}_*\gamma(t)^2 < +\infty$ and $\mathbf{E}_*\beta(t)^2 < +\infty$ for $t = s, \dots, \theta$.
- (ii) The process $(\beta(t), \gamma(t))$ is adapted to the filtration $\{\mathcal{F}_t\}$.
- (iii) For $t = s, \dots, \theta - 1$,

$$X(t+1) - X(t) = \beta(t)(B(t+1) - B(t)) + \gamma(t)(S(t+1) - S(t)).$$

We do not impose additional conditions on strategies such as transaction costs, bid-ask gap, restrictions on short selling; furthermore, we assume that shares are divisible arbitrarily, and that the current prices are available at the time of transactions without delay.

Definition 2.1 *Let $s, q \in \mathbb{Z}$ be such that $s < \theta$. A market model is said to be complete for the time interval $\{s, s+1, \dots, q\}$ if, for any \mathcal{F}_θ -measurable random claim ψ , such that $\mathbf{E}_*\psi^2 < +\infty$, there exists \mathcal{F}_s -measurable initial wealth $X(s)$ and an admissible self-financing strategy defined at the times sequence $\{s, s+1, \dots, q\}$ such that $X(\theta) = B(\theta)B(s)^{-1}\psi$ a.s. (i.e., $B(\theta)B(s)^{-1}\psi$ is replicable with this strategy and this initial wealth).*

Under the assumptions of Definition 2.1, $X(s) = \mathbf{E}_*\psi$, and this is the fair price at time s of an option with the payoff $B(\theta)B(s)^{-1}\psi$ at the expiration time q . This price is uniquely defined, as well as the martingale measure.

The classical Cox-Ross-Rubinstein discrete time model of a single-stock financial market is covered by this definition with $s = 0$ and trivial σ -algebra \mathcal{F}_0 . For this model, $\xi(t)$ takes only two values, $-d_1$ and d_2 , such that $d_k \in (0, 1)$, $k = 1, 2$; see, e.g., [7], Chapter 3. A trivial generalization of the classical Cox-Ross-Rubinstein model gives the following proposition.

Proposition 2.1 *A market model is complete in the sense of Definition 2.1 if $\xi(t)$ takes only two random values, $-d_1$ and d_2 , such that d_k are \mathcal{F}_s -measurable and $d_k \in (0, 1)$ a.s., $k = 1, 2$.*

The pricing of derivatives is usually more difficult for the so-called incomplete market models where a martingale measure is not unique. Some important examples of market incompleteness arise for a modification of the model described above where $d_k(t)$, $k = 1, 2$ are not measurable with respect to \mathcal{F}_{t-1} , i.e., binomial models with dynamically adjusted sizes (i.e., random sizes) of the binary increments; see, e.g., [2]. These binomial models are incomplete.

Let \mathcal{T} be a given subset of \mathbb{Z}^- .

Starting from now, we will consider $t = 0$ as the current time; we will assume that the observations of the prices are available for $t \in \mathcal{T}$. Inevitably, to consider pricing problems for the options expiring at a time $T > 0$, we have to rely on a hypothesis that the properties of the market that we established using the historical observations will somehow be carried forward to the future times $t > 0$. Therefore, we will be considering completeness based on observed prices for negative times.

Theorem 2.1 *Let $\{S(t)\}_{t \in \mathcal{T}}$ be the set of prices for the model described above.*

- (i) *Let $\mathcal{T} = \{t : \theta \leq t \leq 0\}$, for some $\theta < 0$. In this case, for any $\varepsilon > 0$, there exists a market model with the corresponding processes $\{\tilde{S}_\varepsilon(t)\}$ and $\{\xi_\varepsilon(t)\}$ that is complete on the time interval $\{s, \dots, q\}$ for any $s, q \in \mathcal{T}$, $s < q$, and such that*

$$\sup_{t \in \mathcal{T}} (|S_\varepsilon(t) - S(t)| + |\xi_\varepsilon(t) - \xi(t)|) < \varepsilon \quad a.s. \quad (2)$$

- (ii) *Let $\mathcal{T} = \mathbb{Z}^-$ and let there exists $M > 0$ such that $\sum_{t \in \mathcal{T}} (1 + |t|)^{-2M} |\xi(t)|^2 < +\infty$ a.s.. In this case, for any $\tau < 0$ and $\varepsilon > 0$, there exists a market model with the corresponding processes $\{\tilde{S}_\varepsilon(t)\}$ and $\{\xi_\varepsilon(t)\}$ that is complete on the interval $\{s, \dots, q\}$ for any $s, q \in \mathcal{T}$,*

$s < q$, and that

$$\begin{aligned} \sum_{t \in \mathcal{T}} (1 + |t|)^{-2M} (\xi_\varepsilon(t) - \xi(t))^2 &< \varepsilon \quad a.s., \\ \sup_{t: \tau \leq t \leq 0} |S_\varepsilon(t)/S_\varepsilon(\tau) - S(t)/S(\tau)| &< \varepsilon \quad a.s. \end{aligned} \quad (3)$$

Corollary 2.1 *The incomplete markets are indistinguishable from the complete markets in the terms of the market statistics.*

3 Proof of Theorem 2.1

For $r \in [1, +\infty]$ and $\theta, \tau \in \mathbb{Z}$, $\theta \leq \tau$, we denote by $\ell_r(\theta, \tau)$ the Banach space of real valued sequences $\{x(t)\}_{t=\theta}^\tau$ with the norm $\|x\|_{\ell_r(\theta, \tau)} \triangleq (\sum_{t=\theta}^\tau |x(t)|^r)^{1/r}$ for $r < +\infty$ and $\|x\|_{\ell_\infty(\theta, \tau)} \triangleq \sup_t |x(t)|$ for $r = +\infty$. Similar notations will be used for $\theta = -\infty$ and $\tau = +\infty$. In addition, for a $\bar{\mathcal{T}} \subset \mathbb{Z}$, we will use a similar notation $\ell_2(\bar{\mathcal{T}})$ the Banach space of real valued sequences $\{x(t)\}_{t \in \bar{\mathcal{T}}}$ with the norm $\|x\|_{\ell_2(\theta, \tau)} = (\sum_{t=\theta}^\tau |x(t)|^2)^{1/2}$.

Let $\ell_r \triangleq \ell_r(-\infty, +\infty)$.

For $x \in \ell_2$, we denote by $X = \mathcal{Z}x$ the Z-transform

$$X(z) = \sum_{t=-\infty}^{\infty} x(t)z^{-t}, \quad z \in \mathbf{C}.$$

The inverse Z-transform $x = \mathcal{Z}^{-1}X$ is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega t} d\omega, \quad t = 0, \pm 1, \pm 2, \dots$$

We assume that we are given $\Omega \in (0, \pi)$.

For a $\Omega \in (0, \pi)$, let \mathbb{B}^Ω be the set of all mappings $X : \mathbb{T} \rightarrow \mathbf{C}$ such that $X(e^{i\omega}) \in L_2(-\pi, \pi)$ and $X(e^{i\omega}) = 0$ for $|\omega| > \Omega$. We will call the corresponding processes $x = \mathcal{Z}^{-1}X$ *band-limited*. Let ℓ_2^Ω be the set of all band-limited processes from ℓ_2 .

Let $H_\Omega(z)$ be the transfer function for an ideal low-pass filter such that $H_\Omega(e^{i\omega}) = \mathbb{I}_{[-\Omega, \Omega]}(\omega)$, where \mathbb{I} is the indicator function. Let $h_\Omega = \mathcal{Z}^{-1}H_\Omega$.

For a subset $\bar{\mathcal{T}} \subset \mathbb{Z}^-$, let $\ell_2^\Omega(\bar{\mathcal{T}})$ be the subset of $\ell_2(\bar{\mathcal{T}})$ consisting of sequences $\{\hat{x}(t)\}_{t \in \bar{\mathcal{T}}}$ for all $\hat{x} \in \ell_2^\Omega$. We will use notation $\ell_2^\Omega(-\infty, 0) = \ell_2^\Omega(\mathbb{Z}^-)$.

Lemma 3.1 (i) *For any $\tau \in \mathbb{Z}$ and any $\hat{x} \in \ell_2^\Omega(\mathbb{Z}_\tau^-)$, where $\mathbb{Z}_\tau^- \triangleq \{t : t \leq \tau\}$, there exists a unique $x' \in \ell_2$ such that $\hat{x}(t) = x'(t)$ for $t \leq \tau$.*

(ii) *For any $\Omega \in (0, \pi)$, the set $\ell_2^\Omega(-\infty, 0)$ is a closed linear subspace of $\ell_2(-\infty, 0)$.*

(iii) For any $x \in \ell_2(-\infty, 0)$, there exists a unique projection \hat{x}_Ω on $\ell_2^\Omega(-\infty, 0)$. In addition, for $r = 2$ and $r = +\infty$,

$$\|x - \hat{x}_\Omega\|_{\ell_r(-\infty, 0)} \rightarrow 0 \quad \text{as} \quad \Omega \rightarrow \pi - 0.$$

(iv) If \mathcal{T} is a finite set, then $\{x(t)\}_{t \in \mathcal{T}} \in \ell_2^\Omega(\mathcal{T})$ for any $x \in \ell_2$, and there exist more than one $\hat{x}_\Omega \in \ell_2^\Omega$ such that $x(t) = \hat{x}_\Omega(t)$ for $t \in \mathcal{T}$.

Proof of Lemma 3.1. Let us prove statement (i). It suffices to consider $\tau = 0$ only and prove that if $x(\cdot) \in \ell_2^\Omega$ is such that $x(t) = 0$ for $t \leq 0$, then $x(t) = 0$ for $t > 0$. By Theorem 1 from [9], processes $x(\cdot) \in \ell_2^\Omega$ are weakly predictable in the following sense: for any $T > 0$, $\varepsilon > 0$, and $\kappa \in \ell_\infty(0, T)$, there exists $\hat{\kappa}(\cdot) \in \ell_2(0, +\infty) \cap \ell_\infty(0, +\infty)$ such that $\|y - \hat{y}\|_{\ell_2} \leq \varepsilon$, where

$$y(t) \triangleq \sum_{m=t}^{t+T} \kappa(t-m)x(m), \quad \hat{y}(t) \triangleq \sum_{m=-\infty}^t \hat{\kappa}(t-m)x(m).$$

We apply this to a process $x(\cdot) \in \ell_2^\Omega$ such that $x(t) = 0$ for $t \in \mathbb{Z}^-$. Let us observe first that

$$\hat{y}(t) = 0 \quad \forall t < 0. \quad (4)$$

Let $T > 0$ be given. Let us show that $x(t) = 0$ if $0 \leq t \leq T$. Let $\{\kappa_i(\cdot)\}$ be a basis in $\ell_2(-T, 0)$. Let $y_i(t) \triangleq \sum_{m=t}^{t+T} \kappa_i(t-m)x(m)$. It follows from (4) and from the weak predictability [9, 10] of x that $y_i(t) = 0$ if $t \leq 0$. It follows that $x(t) = 0$ if $t \leq T$.

Further, let us apply the proof given above to the process $x_T(t) = x(t+T)$. Clearly, $x_T(\cdot) \in \ell_2^\Omega$ and $x_1(t) = 0$ for $t < 0$. Similarly, we obtain that $x_T(t) = 0$ for all $t \leq T$, i.e., $x(t) = 0$ for all $t < 2T$. Repeating this procedure n times, we obtain that $x(t) = 0$ for all $t < nT$ for all $n \geq 1$. This completes the proof of Lemma 3.1(i). In particular, it follows that there exists $\hat{X} \in \mathbb{B}^\Omega$ such that $\hat{x}(t) = (\mathcal{Z}^{-1}\hat{X})(t)$ for $t \leq 0$.

To prove statement (ii), it suffices to prove that $\ell_2^\Omega(-\infty, 0)$ is a closed linear subspace of $\ell_2(-\infty, 0)$. Consider the mapping $\zeta : \mathbb{B}^\Omega \rightarrow \ell_2^\Omega(-\infty, 0)$ such that $x(t) = (\zeta(X))(t) = (\mathcal{Z}^{-1}X)(t)$ for $t \in \mathbb{Z}^-$. This is a linear continuous operator. By Lemma 3.1(i), it is a bijection. In this case, there exists a unique projection \hat{x} of $\{x(t)\}_{t \in \mathbb{Z}^-}$ on $\ell_2^\Omega(-\infty, 0)$.

Since the mapping $\zeta : \mathbb{B}^\Omega \rightarrow \ell_2^\Omega(-\infty, 0)$ is continuous, it follows that the inverse mapping $\zeta^{-1} : \ell_2^\Omega(-\infty, 0) \rightarrow \mathbb{B}^\Omega$ is also continuous; see, e.g., Corollary in Ch.II.5 [24], p. 77. Since the set \mathbb{B}^Ω is a closed linear subspace of $L_2(-\pi, \pi)$, it follows that $\ell_2^\Omega(-\infty, 0)$ is a closed linear subspace of $\ell_2(-\infty, s)$. This completes the proof of statement (ii).

Let us prove statement (iii). Let $X = \mathcal{Z}(x\mathbb{I}_{\mathbb{Z}^-})$ and $\tilde{X}_\Omega = H_\Omega X$. Clearly,

$$\begin{aligned} \|\hat{x}_\Omega - x\|_{\ell_2(-\infty, 0)} &\leq \|\mathbb{I}_{\mathbb{Z}^-} h_\Omega \circ (x\mathbb{I}_{\mathbb{Z}^-}) - x\mathbb{I}_{\mathbb{Z}^-}\|_{\ell_2(-\infty, 0)} \\ &\leq \text{const} \|\tilde{X}_\Omega(e^{i\omega}) - X(e^{i\omega})\|_{L_2(-\pi, \pi)} \rightarrow 0 \quad \text{as } \Omega \rightarrow \pi. \end{aligned}$$

This completes the proof of statement (iii).

Let us prove statement (iv). Let us select arbitrarily $q \in \mathbb{Z}^- \setminus \mathcal{T}$. Let $\tilde{\mathcal{T}} = \mathcal{T} \cup \{q\}$. Consider a finite system of equations

$$x(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \tilde{X}(e^{i\omega}) e^{i\omega t} d\omega, \quad t \in \tilde{\mathcal{T}}. \quad (5)$$

Let us show that there exists $\tilde{X}(e^{i\omega}) \in L_2(-\Omega, \Omega)$ satisfying this system. Consider a set of linearly independent functions $\{\phi_m\}_{m \in \tilde{\mathcal{T}}}$ from $L_2(-\Omega, \Omega)$ such that

$$\int_{-\Omega}^{\Omega} \phi_m(\omega) e^{i\omega t} d\omega = 0, \quad t \in \tilde{\mathcal{T}} \setminus \{m\}, \quad \int_{-\Omega}^{\Omega} \phi_m(\omega) e^{i\omega m} d\omega \neq 0.$$

In this case, $\tilde{X}(e^{i\omega}) = \sum_{m \in \tilde{\mathcal{T}}}^0 c_m \phi_m(\omega)$ satisfy system (5) if $c_m = \left(\int_{-\Omega}^{\Omega} \phi_m(\omega) e^{i\omega m} d\omega \right)^{-1} x(m)$. Let $X(e^{i\omega}) = \tilde{X}(e^{i\omega})$ for $\omega \in [-\Omega, \Omega]$ and $X(e^{i\omega}) = 0$ for $\omega \in [-\pi, \pi] \setminus [-\Omega, \Omega]$. The process $\hat{x}_\Omega = \mathcal{Z}^{-1}X$ is band-limited and has the desired values $x(t)$ for $t \in \mathcal{T}$. Clearly, these processes \hat{x}_Ω are different for different selections of $x(q)$. This completes the proof of statement (iv) and the proof of Lemma 3.1.

Remark 3.1 *Lemma 3.1(i) implies that the future $\{\hat{x}(t)\}_{t>0}$ of a band-limited process is uniquely defined by its past $\{\hat{x}(t)\}_{t \leq 0}$. This is a reformulation in the deterministic setting of the classical Szegő-Kolmogorov Theorem established for stationary Gaussian processes [22, 23, 17]. Lemma 3.1(iv) implies that if \mathcal{T} is a finite set then any path $\{x(t)\}_{t \in \mathcal{T}}$ is a trace of a band-limited process.*

We now in the position to prove Theorem 2.1. For the case of Theorem 2.1(i), we assume below that $M = 0$.

Let $x(t) \triangleq (1 + |t|)^{-M} |\xi(t)|$, and let $\hat{x}_\Omega(t)$ be the corresponding band-limited process described in Lemma 3.1(iii) if $\mathcal{T} = \mathbb{Z}^+$ or any process described in Lemma 3.1(iv) if \mathcal{T} is finite. By Lemma 3.1(iii),(iv), for any $\varepsilon_1 > 0$, there exists $\Omega = \Omega(\varepsilon_1) \in (0, \pi)$ such that

$$\sup_{t \in \mathcal{T}} |\hat{x}_\Omega(t) - x(t)| = \sup_{t \in \mathcal{T}} |\hat{x}_\Omega(t) - (1 + |t|)^{-M} |\xi(t)|| < \varepsilon_1 \quad \text{a.s.} \quad (6)$$

Let the sign function be defined as $\text{sign}(x) = 1$ for $x \geq 0$ and $\text{sign}(x) = -1$ for $x < 0$.

Consider a market model similar to the one described above and with the stock prices $S_\varepsilon(t)$ such that

$$\begin{aligned}\tilde{S}_\varepsilon(t) &= \tilde{S}_\varepsilon(t-1)(1 + \xi_\varepsilon(t)), \quad \tilde{S}_\varepsilon(t) = B(t)^{-1}S_\varepsilon(t), \quad t \in \mathcal{T}, \\ \tilde{S}_\varepsilon(t) &= \tilde{S}_\varepsilon(t), \quad t < \theta, \quad \mathcal{T} = \{\theta, \dots, 0\} \neq \mathbb{Z}^-, \end{aligned}$$

where

$$\xi_\varepsilon(t) \triangleq \zeta(t)a_\varepsilon(t), \quad \zeta(t) \triangleq \text{sign}(\xi(t)), \quad a_\varepsilon(t) \triangleq (1 + |t|)^M \hat{x}_\Omega(t). \quad (7)$$

Here $\tilde{S}_\varepsilon(t)$ is the discounted price process. The process of bond prices $B(t) > 0$ is such as described above, i.e., it is non-random and such that $B(t+1)/B(t) = \rho$ for some $\rho \geq 1$.

Clearly, for any $s, t \in \mathcal{T}$, $s < t$,

$$\tilde{S}(t) = \tilde{S}(s) \prod_{k=s}^{t-1} (1 + \xi(k+1)), \quad \tilde{S}_\varepsilon(t) = \tilde{S}_\varepsilon(s) \prod_{k=s}^{t-1} (1 + \xi_\varepsilon(k+1)),$$

The process \hat{x}_Ω is band-limited, hence it is predictable in the sense of Lemma 3.1(i). It follows that the process $a_\varepsilon(t)$ is also predictable in the sense of Lemma 3.1(i). Clearly, $|\xi_\varepsilon(t)| = a_\varepsilon(t)$, and the process $|\xi_\varepsilon(t)|$ is also predictable, i.e., $|\xi_\varepsilon(t)|$ is \mathcal{F}_τ -measurable for any $\tau < t \leq 0$. one can select Ω such that (8) holds and that (8) implies (2). Hence the market model with the stock price $S_\varepsilon(t)$ and the bond price $B(t)$ is complete in the sense of Definition 2.1

Let $\varepsilon > 0$ and $\tau < 0$ be given; we assume that $\tau = \theta$ under the assumptions of Theorem 2.1(i). Clearly, there exist $\varepsilon_1 = \varepsilon_1(\varepsilon, \tau) > 0$ and $\Omega = \Omega(\varepsilon_1)$ such that (3) and (6) hold and

$$\begin{aligned} \sup_{t: \tau \leq t \leq 0} |\xi_\varepsilon(t) - \xi(t)| &\leq \varepsilon, \\ \sup_{t: \tau \leq t \leq 0} \left| \prod_{k=s}^{t-1} (1 + \xi_\varepsilon(k+1)) - \prod_{k=s}^{t-1} (1 + \xi(k+1)) \right| &\leq \varepsilon. \end{aligned} \quad (8)$$

Then (3) follows. This completes the proof of Theorem 2.1.

Remark 3.2 *The predictability of band-limited processes used in the proof of Theorem 2.1 does not require optimality of the projection \hat{x} . For example, one can use an ideal low-pass filter applied to x arbitrarily extended on $t > 0$. Furthermore, filters with the exponential energy decay also transfers processes into predictable ones [9]. Therefore, these filters with the exponential energy can be used in the proof of Theorem 2.1 instead of the low-pass filters.*

4 Discussion

Theorem 2.1 leads to a counterintuitive conclusion that the incomplete markets are indistinguishable from the complete markets by econometric methods, i.e., in the terms of the market statistics. Due to rounding errors, the statistical indistinguishability leading to this conclusion cannot be fixed via the sample increasing since the statistics for the incomplete market models can be arbitrarily close to the statistics of the alternative complete models.

It can be elaborated as the following. Assume that we collect the marked data (the sequence of the prices) for $t \leq 0$, with the purpose to test the following hypotheses \mathbf{H}_0 and \mathbf{H}_A about the stock price evolution:

\mathbf{H}_0 : the values $\{|\xi(t)|\}_{t \leq 0}$ do not represent a path of a predictable process (i.e., the market is incomplete); and

\mathbf{H}_A : the values $\{|\xi(t)|\}_{t \leq 0}$ represent a path of a predictable process, i.e. $|\xi(t)|$ are \mathcal{F}_τ -measurable for any $\tau < t$ (i.e., the market is complete).

In these hypotheses, we consider only the properties of the "past" market, leaving aside the speculations about the future properties; this would require additional hypotheses about connections between past observations and the future scenarios.

According to Theorem 2.1, it is impossible to reject hypothesis \mathbf{H}_A based solely on the market prices collected. Due to rounding errors, the statistical indistinguishability leading to this conclusion cannot be fixed via the sample increasing since the statistics for the incomplete market models can be arbitrarily close to the statistics of the alternative complete models. This implies that the commonly accepted selection of an incomplete model is not actually based on the market statistics. However, this selection is justified since it stays in the accordance with general acceptance of the immanent non-predictability of the real world. For instance, we would rather accept a model with the possibility of the unpredictable jumps for the volatility than a model where these jumps can be predicted, even if the statistical data supports both models equally.

Further, it is known that the market completeness is not a robust property: small deviations of the observed binomial prices convert a complete market model into an incomplete one. Thanks to Theorem 2.1 and approximation scheme described above, we can claim now that market incompleteness is also non-robust: small deviations can convert an incomplete model into a complete one. More precisely, it implies that, for any incomplete market from a wide class of models, there exists a complete market model with arbitrarily close discrete sets of the observed prices.

We do not consider approximating models where the values $\xi_\varepsilon(t)$ are predictable, since these models allow and are inconsistent with reasonable systems of market agents' beliefs. In the proofs, we used models where $|\xi_\varepsilon(t)|$ are predicable; these models are arbitrage free and can be consistent with reasonable systems of agents' beliefs.

Unfortunately, the predictability of $|\xi_\varepsilon(t)|$ used in the proof of Theorem 2.1 to set an alternative complete model cannot be applied to option pricing under the "natural" hypothesis \mathbf{H}_0 . The stock returns $\xi(t)$ and $\xi_\varepsilon(t)$ are pathwise close under these hypotheses \mathbf{H}_0 and \mathbf{H}_A for $t \leq 0$; however, their properties are quite different with respect to the predicability, and the future paths of $\xi(t)$ and $\xi_\varepsilon(t)$ will not be necessarily close. Moreover, since the new and the old models produce arbitrarily close sets of prices, an observer, due the rounding error, cannot tell apart these models with certainty, i.e., she cannot tell which model generates the observed data. Effectively, the process $|\xi_\varepsilon(t)|_{t \leq 0}$ in the new model is not observable at time $t = 0$ for an observer from the old model.

It can be noted that we can replace the hypothesis \mathbf{H}_0 by a hypothesis assuming a particular incomplete market model such as a Markov chain model, etc.

References

- [1] Aït-Sahalia, Y., and Mykland, P. (2004). Estimating diffusions with discretely and possibly randomly spaced data: A general theory. *Annals of Statistics* **32**, 2186-2222.
- [2] Akyildirim, E., Dolinsky, Y. Soner, H.M. (2014). Approximating stochastic volatility by recombinant trees. *Annals of Applied Probability* **24**, 2176-2205.
- [3] Andersen, T. G. and Bollerslev, T. (1998). Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. *International Economic Review* **39**, pp. 885-905.
- [4] Andersen, T.G., Bollerslev, T., Diebold, F.X., and Ebens, H. (2001). The distribution of realized stock return volatility. *Journal of Financial Economics* **61**, pp. 4376.
- [5] Andersen, T.G., Bollerslev, T., Diebold, F.X., and Labys, P. (2003). Modeling and forecasting realized volatility. *Econometrica* **71**, pp. 579-625
- [6] Barndorff-Nielsen, O.E., Graversen S.E. and Shephard, N. (2003), Power variation & stochastic volatility: a review and some new results. *Journal of Applied Probability* 41A, 133-143.

- [7] Dokuchaev N.G. *Mathematical finance: core theory, problems, and statistical algorithms*. Routledge, London and New York, January 2007, 209p.
- [8] Dokuchaev, N. (2010). Predictability on finite horizon for processes with exponential decrease of energy on higher frequencies. *Signal processing* **90**, iss. 2, 696–701.
- [9] Dokuchaev, N. (2012). On sub-ideal causal smoothing filters. *Signal Processing* **92**, iss. 1, 219–223.
- [10] Dokuchaev, N. (2012). Predictors for discrete time processes with energy decay on higher frequencies. *IEEE Transactions on Signal Processing* **60**, No. 11, 6027–6030.
- [11] Dokuchaev, N. (2012). On predictors for band-limited and high-frequency time series. *Signal Processing* **92**, iss. 10, 2571–2575.
- [12] Dokuchaev, N. (2012). On statistical indistinguishability of the complete and incomplete markets, preprint, arXiv:1209.4695.
- [13] Dokuchaev, N. (2014). On strong causal binomial approximation for stochastic processes. *Discrete and Continuous Dynamical Systems – Series B (DCDS-B)* **20**, No.6, 1549–1562.
- [14] Elliott, R.J., Hunter, W.C., and Jamieson, B.M. (1998). Drift and volatility estimation in discrete time. *Jour. of Economic Dynamics & Control* **22**, 209–218.
- [15] Guasoni, P. and Rásonyi, M. (2012). Fragility of arbitrage and bubbles in diffusion models. Working paper, <http://ssrn.com/abstract=1856223>.
- [16] Hull, J., and White, A. (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance* **42**, 381–400.
- [17] Kolmogorov, A.N. (1941). Interpolation and extrapolation of stationary stochastic series. *Izv. Akad. Nauk SSSR Ser. Mat.*, 5:1, 3–14.
- [18] Madan D.B. (1983). Inconsistent Theories as Scientific Objectives. *Philosophy of Science*, Vol. 50, No. 3, pp. 453–470.
- [19] Madan, D.B., and Eberlein, E. (2012). Dealing with complex realities in financial modeling. *Current science* **103** (6), 647–649.
- [20] Malliavin, P., and Mancino, M.E. (2002). Fourier Series method for measurement of multivariate volatilities. *Finance & Stochastics* **6**, 49–62.

- [21] Pliska, S. R. (1997). *Introduction to mathematical finance: discrete time models*. Blackwell Publishers, Oxford, UK, and Malden, Mass.
- [22] Szegő, G. (1920). Beiträge zur Theorie der Toeplitzschen Formen. *Math. Z.* 6, 167–202.
- [23] Szegő, G. (1921). Beiträge zur Theorie der Toeplitzschen Formen, II. *Math. Z.* 9, 167-190.
- [24] Yosida, K. (1965). *Functional Analysis*. Springer, Berlin Heilderberg New York.